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Openness of the Metric Projection in Certain Banach Spaces

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Let E be a real Banach space and K a nonempty closed convex subset of E. The *metric projection* (or nearest-point mapping) P_K of E onto K is defined (when it exists) by

$$\|x - P_K x\| = \inf\{\|x - y\| : y \in K\}, \qquad x \in E.$$

This will exist and be single valued, for instance, whenever E is a reflexive space with rotund (strictly convex) norm. We will usually write P for P_K. Throughout the paper we will assume that K *has nonempty interior which* (without loss of generality) *contains the origin.* This simplifies the description of the *Minkowski functional* (or gauge) μ_K associated with K:

$$\mu_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}, \qquad x \in E.$$

We will frequently write μ in place of μ_K.

Most of the results in this paper were originally proved in Hilbert space, while trying to understand better the relationship between differentiability of μ (at nonzero points) and differentiability of P in $E \backslash K$. For higher-order Fréchet differentiability, this is well understood (see [6, 9, 10]): μ being C^{k+1} is essentially equivalent to P being C^k. Even in two-dimensional Euclidean space, however, there are examples where μ is C^1 but P is not everywhere differentiable in $E \backslash K$, or where P is C^1 but μ has nonzero points of nondifferentiability. Take for instance, K to be the epigraph of $y = x^{4/3} - 1$ or of $y = |x| + x^{4/3} - 1$, respectively. (The proofs are not immediate.) What we show below is that differentiability of μ corresponds to *openness* of P, a fact which will be seen to be unsurprising when viewed from the right perspective.

DEFINITION. The metric projection P_K is said to be *open* [*weakly open*]

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provided the image of each open [weakly open] subset of $E\backslash K$ is a relatively norm-open subset of the boundary bdry K of K.

Our proofs will make considerable use of the well-known duality mapping.

DEFINITION. A *duality mapping* for the Banach space E is a map $J: E \to E^*$ which satisfies $\|J(x)\| = \|x\|$ and $\langle J(x), x\rangle = \|x\|^2$ for each $x \in E$.

In Hilbert space, J is just the identity mapping. In general Banach spaces, the existence of at least one such map is guaranteed by the Hahn–Banach theorem. It will be uniquely determined precisely when E is *smooth*, that is, when the norm in E is Gateaux differentiable at each nonzero point. It will be one–one precisely when E is rotund. In a smooth space, J is always norm-to-norm continuous precisely when the norm in E is Fréchet differentiable (away from the origin).

DEFINITION. The *subdifferential* $\partial\mu_K(x)$ of μ_K at the point x is the set of all x^* in E^* satisfying $\langle x^*, y\rangle \leqslant \mu_K(y)$ for all $y \in E$ and $\langle x^*, x\rangle = \mu_K(x)$.

The Hahn–Banach theorem guarantees that $\partial\mu_K(x)$ is nonempty. Observe that the mapping J is simply (within a scalar multiple) a selection for the subdifferential of the norm.

Differentiability of μ_K can be characterized in terms of the subdifferential, as follows: *The function μ_K is Gateaux differentiable at the point x if and only if $\partial\mu_K(x)$ consists of a single point*, which we will denote by $d\mu_K(x)$. If this be the case, then $\partial\mu_K$ is norm-to-weak* upper semicontinuous at x, that is, if $\|x_n - x\| \to 0$ and if $x_n^* \in \partial\mu_K(x_n)$, then $x_n^* \to d\mu_K(x)$ weak*. *The function μ_K is Fréchet differentiable at x if and only if it is Gateaux differentiable there and $\|x_n^* - d\mu_K(x)\| \to 0$ whenever $\|x_n - x\| \to 0$* and $x_n^* \in \partial\mu_K(x)$, that is, $\partial\mu_K$ is norm-to-norm upper semicontinuous at x.

That these characterizations are valid for arbitrary continuous convex functions is well known; see, for instance, Giles [7] (where a selection like J is called a *support mapping*). The basic facts about Banach spaces which we use may be found in Day [3] or Diestel [4].

It follows readily from the definition of P that if $x \in E\backslash K$ and $z = Px$, then $z + \mathbb{R}^+(x-z) \subseteq P^{-1}z$ so that $P^{-1}z$ is the union of all such rays. The set $P^{-1}z$ will be nonempty for every z in bdry K if E is reflexive; this well-known fact is implicit in the following lemma and, when valid for all K in E, implies that E is reflexive. The lemma describes the set $\partial\mu_K(y)$ in terms of J and $P_K^{-1}y$; it says that the former is the normalized image under J of (a translate of) the inverse image of P_K.

1. LEMMA. *Suppose that E is smooth and reflexive, and that $y \in$* bdry K. *Then $\partial\mu(y) = \{\langle J(x-Px), Px\rangle^{-1} J(x-Px) : x \in E\backslash K$ and $Px = y\}$.*

Proof. Suppose, first, that $x^* \in \partial\mu(y)$. Since $\langle x^*, y\rangle = 1 = \mu(y)$, x^* is nonzero. By reflexivity, we can choose $z \in E$, $\|z\| = 1$, such that $\langle x^*, z\rangle = \|x^*\|$. Let $x = y + z$ and observe that $\|x^*\|^{-1} x^* = J(z) = J(x - y)$, so that $\langle J(x-y), y\rangle^{-1} J(x-y) = x^*$. Thus, it suffices to prove that $x \in E\backslash K$ and that $y = Px$. Since $\mu(x) \geqslant \langle x^*, x\rangle = \langle x^*, y+z\rangle = 1 + \|x^*\| > 1$ and $\sup \mu(K) = 1$, the former is clear. For the latter, note that for all $u \in K$, we have $\langle x^*, u\rangle \leqslant \mu(u) \leqslant 1$ and hence

$$\|x^*\| \cdot \|x - u\| \geqslant \langle x^*, x - u\rangle = 1 + \|x^*\| - \langle x^*, u\rangle \geqslant \|x^*\|,$$

so that $\|x - y\| = 1 \leqslant \|x - u\|$ for all such u.

To prove the reverse inclusion, suppose that $x \in E\backslash K$ and $Px = y$. By the separation theorem (applied to K and the ball of radius $\|x - Px\|$ centered at x), there exists $y^* \in E^*$, $\|y^*\| = 1$, such that

$$\begin{aligned} \sup\{\langle y^*, u\rangle : u \in K\} &= \langle y^*, y\rangle \\ &= \inf\{\langle y^*, u\rangle : u \in B\} = \langle y^*, x\rangle - \|x - Px\|. \end{aligned} \tag{1}$$

Thus, $\langle y^*, x - y\rangle = \langle y^*, x - Px\rangle = \|x - Px\|$; it follows from smoothness that $\|x - Px\| y^* = J(x - Px)$. Let $x^* = \langle J(x - Px), Px\rangle^{-1} J(x - Px)$; it remains to show that $x^* \in \partial\mu(y)$. By routine arguments, this reduces to showing that $\langle x^*, y\rangle \leqslant 1$ for all $u \in K$, with equality when $u = y$. Since $x^* = \langle y^*, y\rangle^{-1} y^*$, this is immediate from (1).

Another interpretation of this lemma is that (modulo the duality mapping J) the maps $\partial\mu$ and P^{-1} (suitably normalized) are almost the same. Thus, it should not be surprising if openness of P (roughly, continuity of P^{-1}) should correspond to continuity of $\partial\mu$, which is essentially differentiability of μ. Our first theorem will make these remarks precise for the case of Gateaux differentiability. Note that the positive homogeneity of μ implies that it is differentiable at a point x if and only if it is differentiable at λx, for every $\lambda > 0$. Since μ will not be differentiable at points where it equals zero, we will restrict our attention to those points where it equals one, namely, to bdry K. In the following theorem we will use the well-known fact that a continuous convex function on a reflexive Banach space is Gateaux (in fact, Fréchet) differentiable at a dense set of points. This is due to Asplund [1] and has been extended in a number of directions; see [11] and the expositions in Bourgin [2], Diestel–Uhl [5] or Giles [7].

2. THEOREM. *If E is a rotund reflexive space and if P_K is weakly open, then μ_K is Gateaux differentiable at each point of* bdry K.

Proof. Suppose that $z \in$ bdry K and that μ is not Gateaux differentiable at z. This is equivalent to saying that $\partial\mu(z)$ contains distinct points x_1^*, x_2^*, so by Lemma 1 there exist x_1, x_2 in $E\backslash K$ such that $P(x_i) = z$, $\|x_i - z\| = 1$

and $x_i^* = \langle J(x_i - z), z\rangle^{-1} J(x_i - z)$, $i = 1, 2$. It is clear that $x_1 \neq x_2$. Choose $0 < \varepsilon < 1/4$ such that $\varepsilon < 4^{-1}(1 + 4\|x_1 - x_2\|)^{-1}\|x_1 - x_2\|^2$ and define weakly open neighborhoods W_1, W_2, of x_1, x_2, respectively, by setting $W_i = F_i \cap H_i \cap G_i$, where

$$H_i = \{u \in E: \langle J(x_i - z), u - z\rangle > 1 - \varepsilon\}, \qquad i = 1, 2,$$

$$F_1 = \{u \in E: \langle J(x_2 - z), u - z\rangle < 1 + \varepsilon\},$$

$$F_2 = \{u \in E: \langle J(x_1 - z), u - z\rangle < 1 + \varepsilon\},$$

$$G_1 = \{u \in E: \langle J(x_2 - x_1), u - x_1\rangle < \varepsilon\},$$

$$G_2 = \{u \in E: \langle J(x_2 - x_1), u - x_1\rangle > \|x_2 - x_1\|^2 - \varepsilon\}.$$

These are all weakly open and it is easily verified that $x_i \in W_i$, $i = 1, 2$. That $W_i \subseteq E\backslash K$ is seen by recalling that for $u \in K$ we have $1 \geqslant \mu(u) \geqslant \langle x^*, u\rangle = \langle J(x_i - z), z\rangle^{-1}\langle J(x_i - z), u\rangle$, hence $\langle J(x_i - z), u - z\rangle \leqslant 0$ and therefore $u \notin H_i$, $i = 1, 2$. Now, by hypothesis, both $P(W_1)$ and $P(W_2)$ are relatively open in bdry K, hence $U = P(W_1) \cap P(W_2)$ is a relatively open neighborhood of z. Since E is reflexive, there exists $y \in$ bdry K such that $\|y - z\| < \varepsilon$ and μ is Gateaux differentiable at y. There exist $u_i \in W_i$, $i = 1, 2$, such that $Pu_1 = Pu_2 = y$. By Lemma 1, we conclude that $\langle J(u_i - y), y\rangle^{-1} J(u_i - y) \in \partial\mu(y)$, $i = 1, 2$. Since μ is Gateaux differentiable at y, these functionals coincide, so if $\lambda = \langle J(u_1 - y), y\rangle^{-1}\langle J(u_2 - y), y\rangle$, then $J(u_2 - y) = \lambda J(u_1 - y) = J(\lambda(u_1 - y))$; by rotundity of E, $u_2 - y = \lambda(u_1 - y)$. We will show that this implies that $u_2 \notin G_2$, a contradiction which will complete the proof. We need some estimates for λ. Since $u_2 \in H_2$, $\|y - z\| < \varepsilon$ and $u_1 \in F_1$, we have

$$\begin{aligned} 1 - \varepsilon &< \langle J(x_2 - z), u_2 - z\rangle = \langle J(x_2 - z), u_2 - y\rangle + \langle J(x_2 - z), y - z\rangle \\ &< \lambda\langle J(x_2 - z), u_1 - y\rangle + \varepsilon < \lambda\langle J(x_2 - z), u_1 - z\rangle + \lambda\varepsilon + \varepsilon \\ &< \lambda(1 + \varepsilon) + \lambda\varepsilon + \varepsilon \end{aligned}$$

so $\lambda > (1 + 2\varepsilon)^{-1}(1 - 2\varepsilon)$ and therefore $1 - \lambda < (1 + 2\varepsilon)^{-1} 4\varepsilon$. A similar argument, using $u_2 \in F_2$, $\|y - z\| < \varepsilon$ and $u_1 \in H_1$, shows that $\lambda < (1 - 2\varepsilon)^{-1}(1 + 2\varepsilon) < 3$ (since $\varepsilon < 1/4$) and hence $1 - \lambda > (1 - 2\varepsilon)^{-1}(-4\varepsilon)$. It follows that $|1 - \lambda| < 8\varepsilon$. Our earlier restriction on ε will imply that $u_2 \notin G_2$, that is,

$$\langle J(x_2 - x_1), u_2 - x_1\rangle \leqslant \|x_1 - x_2\|^2 - \varepsilon.$$

To see this, write $u_2 = \lambda u_1 + (1 - \lambda)y$, so that

$$\begin{aligned} \langle J(x_2 - x_1), u_2 - x_1\rangle &= \lambda\langle J(x_2 - x_1), u_1 - x_1\rangle \\ &\quad + (1 - \lambda)\langle J(x_2 - x_1), y - x_1\rangle. \end{aligned}$$

The first term is at most 3ε. To estimate the second term, observe that

$$\langle J(x_2 - x_1), y - x_1\rangle = \langle J(x_2 - x_1), z - x_1\rangle + \langle J(x_2 - x_1), y - z\rangle$$

has modulus at most $(1+\varepsilon)\|x_1 - x_2\| < 2\|x_1 - x_2\|$. Since $|1-\lambda| < 8\varepsilon$, the second term is bounded in modulus by $16\varepsilon\|x_1 - x_2\|$.

Thus, we will have the desired contradiction provided $3\varepsilon + 16\varepsilon\|x_1 - x_2\| \leqslant \|x_1 - x_2\|^2 - \varepsilon$, an inequality which is guaranteed by our choice of ε.

Our next theorem (which is much simpler) is the "strong" version of the previous one.

3. THEOREM. *Suppose that E is reflexive, with a rotund, Fréchet differentiable norm. If P is open, then μ is Fréchet differentiable at each point of* bdry K.

Proof. Since P is necessarily weakly open, Theorem 2 implies that the Gateaux derivative $d\mu(z)$ exists at each point z of bdry K. It suffices to show that if $\{z_n\} \subseteq \text{bdry}\, K$ and $z_n \to z_0$, then $d\mu(z_n) \to d\mu(z_0)$. From Lemma 1 we can choose $\{x_n\}_{n=0}^{\infty} \subseteq E\backslash K$ such that $\|x_n - z_n\| = 1$, $P(x_n) = z_n$ and

$$d\mu(z_n) = \langle J(x_n - z_n), z_n\rangle^{-1} J(x_n - z_n), \qquad n = 0, 1, 2, \ldots.$$

Since P is open, given $\varepsilon > 0$ there exist $N \geqslant 1$ and points $y_n \in E\backslash K$ such that $\|y_n - x_0\| < \varepsilon$ and $Py_n = z_n$ for $n \geqslant N$. By Lemma 1, again, for each such n we must have

$$\langle J(y_n - z_n), z_n\rangle^{-1} J(y_n - z_n) = d\mu(z_n).$$

Since J is one–one, this implies that there exists $\lambda_n > 0$ such that $y_n = z_n + \lambda_n(x_n - z_n)$. Thus, $\|y_n - z_n\| = \lambda_n$ and $\|y_n - x_n\| = |1 - \lambda_n|$. Also,

$$|\lambda_n - 1| = |\|y_n - z_n\| - \|x_0 - z_0\|| \leqslant \|y_n - x_0\| + \|z_n - z_0\|$$

so

$$\|x_n - x_0\| \leqslant \|x_n - y_n\| + \|y_n - x_0\| \leqslant 2\varepsilon + \|z_n - z_0\| \qquad \text{for} \quad n \geqslant N.$$

Since $\|z_n - z_0\| \to 0$, this implies that $x_n \to x_0$ and hence $x_n - z_n \to x_0 - z_0$. Since J is norm continuous, we have $J(x_n - z_n) \to J(x_0 - z_0)$ and

$$\begin{aligned}&|\langle J(x_n - z_n), z_n\rangle - \langle J(x_0 - z_0), z_0\rangle| \\ &\qquad \leqslant |\langle J(x_n - z_n), z_n - z_0\rangle| + |\langle J(x_n - z_n) - J(x_0 - z_0), z_0\rangle| \to 0\end{aligned}$$

so that $d\mu(z_n) \to d\mu(z_0)$.

Note that if K is the unit ball of a rotund Banach space E, then for any x

in $E\backslash K$ we have $P_K x = \|x\|^{-1} x$. It is easily seen that this is always open, even when μ_K (in this case, the norm of E) has no special smoothness properties. Thus the theorem above is obviously invalid without the hypothesis that the norm in E be Fréchet differentiable.

4. LEMMA. *Suppose that E is reflexive, smooth and rotund, and that J^{-1} is continuous. If $x \in E\backslash K$ and if μ is Fréchet differentiable at Px, then for any sequence $\{z_n\} \subseteq \operatorname{bdry} K$ such that $\|z_n - Px\| \to 0$ there exists a sequence $\{x_n\} \subseteq E\backslash K$ with $Px_n = z_n$ and $\|x_n - x\| \to 0$.*

Proof. Since E is reflexive we can choose $\{y_n\} \subseteq E\backslash K$ such that $Py_n = z_n$ for all n. Let $r_n = \langle J(y_n - z_n), z_n\rangle^{-1} \langle J(x - Px), Px\rangle$ (each of these is positive) and let $x_n = z_n + r_n(y_n - z_n)$. Then $Px_n = z_n$ for each n and hence, by Lemma 1, $\langle J(x_n - z_n), z_n\rangle^{-1} J(x_n - z_n) \in \partial\mu(z_n)$. Since $z_n \to Px$ and μ is Fréchet differentiable at Px, this sequence of functionals converges in norm to $d\mu(Px) = \langle J(x - Px), Px\rangle^{-1} J(x - Px)$. Now, $x_n - z_n = r_n(y_n - z_n)$ so for each n, $\langle J(x_n - z_n), z_n\rangle = r_n \langle J(y_n - z_n), z_n\rangle = \langle J(x - Px), Px\rangle$. This implies that $J(x_n - z_n) \to J(x - Px)$ and by continuity of J^{-1}, that $x_n - z_n \to x - Px$, so $x_n \to x$.

This result makes it easy to prove a converse to Theorem 3.

5. COROLLARY. *Suppose that E is smooth and that the norm in E^* is Fréchet differentiable. If μ_K is Fréchet differentiable at each point of* bdry K, *then P_K is open.*

Proof. Suppose that U is a nonempty open subset of $E\backslash K$. It suffices to show that if $z \in P(U)$ and if $\{z_n\} \subseteq \operatorname{bdry} K$ with $z_n \to z$, then there exists $\{x_n\} \subseteq E\backslash K$ with $Px_n = z_n$ and $\{x_n\}$ eventually in U. This is immediate from Lemma 4 once we observe that the hypotheses are fulfilled. But Fréchet differentiability of the norm in E^* implies that E is reflexive [3], that J^{-1} is continuous (it is, after all, the differential of the dual norm) and that E is rotund.

The hypotheses for the following converse to Theorem 2 are obviously satisfied if E is a Hilbert space.

6. PROPOSITION. *Suppose that E is reflexive, smooth and rotund and that the duality map J is weakly sequentially continuous. If μ_K is Gateaux differentiable at nonzero points, the P_K is weakly open.*

Proof. We use essentially the same method of proof as the preceding result. Suppose that W is a nonempty weakly open subset of $E\backslash K$ and that $z \in P(W)$ (so that $z = Px$ for some $x \in W$). If $\{z_n\} \subseteq \operatorname{bdry} K$ with $z_n \to z$, it suffices to produce $\{x_n\} \subseteq E\backslash K$ with $Px_n = z_n$ such that $\{x_n\}$ converges weakly to x. To this end we define $\{x_n\}$ exactly as in the proof of Lemma 4 and, using the hypothesis that $d\mu$ is norm-weak continuous, we can conclude that the bounded sequence $\{J(x_n - z_n)\}$ converges weakly to $J(x - Px)$. From the weak relative sequential compactness of bounded subsets of E and the continuity hypothesis on J it follows that $\{x_n - z_n\}$ converges weakly to $x - Px$, which yields the desired conclusion.

It appears to be a difficult problem to determine those rotund, smooth and reflexive Banach spaces for which the duality mapping J is weakly sequentially continuous. It is trivially the case for smooth and rotund finite dimensional spaces and for l_2. Y. Benyamini has shown us how these examples can be combined to produce separable examples which are not isomorphic to l_2: Take E to be the l_2 sum of a sequence of smooth and rotund spaces $\{E_n\}$ such that $\dim E_n = n$ and such that the Banach–Mazur distance between E_n and $l_2^{(n)}$ tends to infinity with n. (For instance, let $E_n = l_4^{(n)}$.) On the other hand, if $2 < p < \infty$, then the l_p sum of l_2 and $\mathbb{R}$ is isomorphic to l_2 but the duality map is *not* weakly sequentially continuous.

A different characterization of the differentiability of μ in terms of P was obtained for Hilbert space in [6, Proposition 3.3 and 3.4]. It essentially says that Gateaux (or Fréchet) differentiability of μ at a point $x \in \operatorname{bdry} K$ is equivalent to P having the identity map for the tangent hyperplane $T[x] = \{u\colon \langle d\mu(x), u\rangle = 0\}$ as its *partial Gateaux* (or *Fréchet*) *derivative* at x. That is, for $u \in T[x]$,

$$P(x + tu) = x + tu + o(t) \qquad \text{(Gateaux case)}$$

or

$$P(x + u) = x + u + o(u) \qquad \text{(Fréchet case).}$$

(Note that since $Px = x$, these are indeed assertions about the partial differentiability of P at x.) As we will show below, these results are valid in spaces much more general than Hilbert space, but we must first extend some basic lemmas to Banach spaces. The first of these is a well-known characterization of nearest points.

7. LEMMA. *Suppose that E is a smooth Banach space and that C is a closed nonempty convex subset of E. If $x \in E\backslash C$ and if $P_C x$ exists, then it satisfies the "defining inequality"*

$$\langle J(x - Px), u - Px\rangle \leqslant 0 \qquad \textit{for all } u \in C. \tag{$*$}$$

The nomenclature is based on the fact that if z is any element of C satisfying $(*)$ (with z in place of Px), then z is a nearest point in C to x.

Proof. The proof that $P_C x$ satisfies $(*)$ is identical to the second half of the proof of Lemma 1 (which did not require the reflexivity hypothesis); the assertion there that $\langle x^*, u\rangle \leqslant 1$ for $u \in K$, is equivalent to

$$\langle J(x-Px), Px\rangle^{-1}\langle J(x-Px), u > \leqslant 1,$$

which is (replacing K by C) condition $(*)$. Suppose, then, that the point $z \in C$ is such that $\langle J(x-z), u-z\rangle \leqslant 0$ for all $u \in C$. Then

$$\begin{aligned} 0 &\geqslant \langle J(x-z), u-z\rangle = \langle J(x-z), u-x+x-z\rangle \\ &= \|x-z\|^2 + \langle J(x-z), u-x\rangle \end{aligned}$$

so

$$\|x-z\|^2 \leqslant \langle J(x-z), x-u\rangle \leqslant \|x-z\| \cdot \|x-u\|$$

or

$$\|x-z\| \leqslant \|x-u\| \qquad \text{for all } u \in C, \text{ that is, } z \text{ is a nearest point in } C \text{ to } x.$$

We next recall the definition of a well-known object.

DEFINITION. If C is closed and convex and if $x \in C$, the *support cone* $S_C(x)$ to C at x is the closure of the convex cone $\bigcup \{\lambda(C-x): \lambda > 0\}$.

The set $S_C(x)$ is obviously a closed convex cone with vertex 0; it is the smallest such cone S whose translate $x+S$ has vertex x and contains C (hence the terminology). It enters into the formulation of a basic result concerning directional derivatives of P_C at points of C. This has been proved by Zarantonello [14, Lemma 2.3] for Hilbert space, but his proof does not require an inner product. We include a shortened version of his proof, since the result will be applied below.

8. LEMMA (Zarantonello). *Suppose that C is closed and convex, that P_C exists and that $x \in C$. For all $u \in S_C(x)$ we have*

$$P_C(x+tu) = x + tu + o(t), \qquad t > 0.$$

Proof. There is no loss in generality in assuming that $x = 0$. By definition if $u \in S_C(0)$, there exist sequences $\{x_n\} \subseteq C$ and $t_n > 0$ such that $t_n^{-1}x_n \to u$. Fixing n for the moment, suppose that $0 < t \leqslant t_n$. Then $0 < tt_n^{-1} \leqslant 1$ so $tt_n^{-1}P(t_n u) \in C$ and hence

$$\begin{aligned} t^{-1}\|tu - P(tu)\| &\leqslant t^{-1}\|tu - tt_n^{-1}P(t_n u)\| \\ &= t_n^{-1}\|t_n u - P(t_n u)\| \\ &\leqslant t_n^{-1}\|t_n u - x_n\| = \|u - t_n^{-1}x_n\|. \end{aligned}$$

Thus, $\limsup_{t\to 0+} \|tu - P(tu)\| \leqslant \|u - t_n^{-1}x_n\|$, for each n. Since the right side converges to 0, this completes the proof.

(More general versions of this lemma have been proved for Hilbert space by Haraux [8], Mignot [12] and Zarantonello [14], and for certain reflexive Banach spaces in [13].)

Our first application of these notions generalizes [6, Proposition 3.3]. Note that the hypotheses on P are satisfied if E is rotund and reflexive.

9. Proposition. *Suppose that P_K exists and is single valued and that $x \in \operatorname{bdry} K$. If μ_K is Gateaux [Fréchet] differentiable at x, then P_K has the identity map for $T[x] = \{u\colon \langle d\mu(x), u\rangle = 0\}$ as its partial Gateaux [Fréchet] derivative at x.*

Proof. An application of the separation theorem shows that, if $d\mu(x)$ exists, then the support cone $S_K(x)$ necessarily coincides with the half-space $\{u \in E\colon \langle d\mu(x), u\rangle \leqslant 0\}$ bounded by $T[x]$. Thus, if $u \in T[x]$, then $\pm u \in S_K(x)$ and Lemma 8 shows that the identity map in $T[x]$ is the partial Gateaux derivative for P in $T[x]$ at x. The proof for the Fréchet case is identical to that for [6, Proposition 3.3] (noting the misprint in the last line of [6, p. 489], where the brackets should be replaced by absolute values).

For the converse to this proposition we do not, of course, assume that $d\mu$ exists, so we must find an alternative way to describe $T[x]$. This is easily done in a smooth reflexive space: Use the Hahn–Banach theorem and reflexivity to choose, for $x \in \operatorname{bdry} K$, a nonzero element w such that $P_K(x + w) = x$. We then let $T[x] = \{u\colon \langle J(w), u\rangle = 0\}$. (It is readily verified that, within a positive scalar multiple, $J(w)$ is in $\partial\mu(x)$, and hence this definition of $T[x]$ coincides with the previous one whenever $d\mu(x)$ exists.)

10. Proposition. *Suppose that E is reflexive, smooth and rotund, and that $x \in \operatorname{bdry} K$. Let $w \neq 0$ be such that $P(x + w) = x$ and suppose that the identity map for $T[x] = \{u\colon \langle J(w), u\rangle = 0\}$ is the partial Gateaux [Fréchet] derivative at x for P_K in $T[x]$. Then μ is Gateaux [Fréchet] differentiable at x.*

The proof of this result in [6] makes no use of the inner product, so there is no need to revise it, other than to replace w by $J(w)$ whenever the former appears in the role of a linear functional. The rotundity of E is used to guarantee that P is single valued.

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